OA3602

Search Theory and Detection

Discrete Random Coverage Models

Objectives:
- Discrete random coverage
- Derivations
Discrete random coverage

Formulation and assumptions

- For discrete random coverage:
  - independent observations
  - bivariate (2D) uniform
  - single cell coverage per time step
  - no excess coverage

(MATLAB code available on course site)
Discrete random coverage ratio

Bottom line, up front

Coverage factor for random coverage

- Coverage ratio for discrete random coverage

\[ F(n) = 1 - \left(1 - \frac{1}{C_S}\right)^n \]

- Recall coverage ratio for discrete complete coverage:

\[ F_{\text{ideal}}(n) = \frac{n}{C_S} \]
Discrete random coverage model

*Derivations*

• Consider similar derivation process as for continuous random coverage:

\[
C(n + 1) = C(n) + \left( \begin{array}{c} \# \text{ new cells covered} \\ \text{in current time step} \end{array} \right)
\]

where, for (uniform) random coverage, we have

Such that we get:
Recall the definition of the coverage ratio for discrete coverage:

\[
F(n) \triangleq \frac{C(n)}{C_S} \quad \Rightarrow \quad F(n + 1) = F(n) + [1 - F(n)] \cdot \frac{1}{C_S}
\]

\[
= \frac{1}{C_S} + \left[1 - \frac{1}{C_S}\right] F(n)
\]

For succinctness, consider letting \( x \triangleq 1 - \frac{1}{C_S} \)

\[
F(n + 1) = \frac{1}{C_S} + xF(n)
\]
Discrete random coverage model

Derivations, continued

\[ F(n + 1) = \frac{1}{C_S} + xF(n) \]

We’d like to get an expression where \( F(n) \) is only on one side of the “=“ sign

Apply the recursion, i.e., iterate counter \( n \)

\[ F(0) = 0 \]

\[ F(1) = \]

\[ F(2) = \]

\[ F(3) = \]

\[ \vdots \]

\[ F(n) = \frac{1}{C_S} \left( 1 + x + x^2 + \cdots + x^{n-1} \right) = \frac{1}{C_S} \sum_{k=0}^{n-1} x^k \]
Discrete random coverage model

Derivations, continued

Making use of the \textit{finite power series identity}:

\[
\sum_{k=0}^{n-1} x^k = \frac{1 - x^n}{1 - x}
\]

the previous expression takes the form:

Substituting back our variable, \(x \triangleq 1 - \frac{1}{C_S}\), we get:

\[
F(n) = 1 - \left(1 - \frac{1}{C_S}\right)^n
\]

Coverage ratio for discrete random coverage
Discrete random coverage model

Limiting behaviors

Initial condition: \( F(0) = 0 \) (zero initial coverage)

Time till complete coverage: \( n^* \)

\[
1 = F(n^*) = 1 - \left(1 - \frac{1}{C_S}\right)^{n^*}
\]

\[
0 = \left(1 - \frac{1}{C_S}\right)^{n^*}
\]

\[
-\infty = \log \left(1 - \frac{1}{C_S}\right)^{n^*}
\]

\[
= n^* \log \left(1 - \frac{1}{C_S}\right)
\]

\[\Rightarrow n^* = +\infty \] (asymptotic time)

Limiting coverage ratio: \( F(n) \)

\[
\lim_{n \to \infty} F(n) = \lim_{n \to \infty} \frac{1}{C_S} \sum_{k=0}^{n-1} x^k
\]

\[
= \frac{1}{C_S} \cdot \sum_{k=0}^{\infty} x^k
\]

\[
= \frac{1}{C_S} \cdot \frac{1}{1 - x}
\]

\[
= \frac{1}{C_S} \cdot \frac{1}{1/C_S}
\]

\[
\lim_{n \to \infty} F(n) = 1
\] (complete coverage)

Infinite power series identity: \( \sum_{k=0}^{\infty} x^k = \frac{1}{1 - x} \)
Discrete random coverage ratio

Recall the time it takes for complete coverage for the ideal case:

\[(n^*)_{\text{ideal}} = C_S\]

\[F_{\text{ideal}}(n^*) = 1\]

The value of coverage ratio at the same time is:

\[F_{\text{random}}(n^*) = 1 - \left(1 - \frac{1}{C_S}\right)^{C_S}\]

\[= 0.6513 \quad \text{(for } C_S=10\text{)}\]
Parting shots

• Today we learned...
  – Discrete random coverage

• Next time...
  – Summary of random coverage
  – Rate-of-change of coverage

Don’t forget to turn in your mud cards!