Proof that \( \mathbb{E}[T] = \int_0^\infty [1 - F(t)] \, dt \)

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Claim: \( \mathbb{E}[T] = \int_0^\infty [1 - F(t)] \, dt \)

Note that the conditions for the above statement is that \( T \) is a proper, non-negative random variable.

Proof:

Recall that the cumulative distribution function is defined as
\[
F(t) = \int_0^t f(s) \, ds \quad \Rightarrow \quad 1 - F(t) = \int_t^\infty f(s) \, ds
\]

Substitution of the above expression into the statement of the claim is:
\[
\int_0^\infty [1 - F(t)] \, dt = \int_t^\infty \left( \int_s^\infty f(s) \, ds \right) \, dt
\]

Changing the order of integration requires that the limits on the \( t \) variable range from 0 to \( s \), and for \( s \) to range between 0 and \( \infty \):
\[
\int_t^\infty \left( \int_s^\infty f(s) \, ds \right) \, dt = \int_s^\infty \int_t^s f(s) \, dt \, ds
\]
\[
= \int_s^\infty f(s) \left( \int_t^s dt \right) \, ds
\]
\[
= \int_s^\infty f(s) \left( t \bigg|_{t=0}^{t=s} \right) \, ds
\]
\[
= \int_s^\infty f(s)s \, ds
\]
\[
= \int_t^\infty f(t) \, dt
\]

Finally, recalling that
\[
\mathbb{E}[T] \overset{\text{def}}{=} \int_0^\infty t \, f(t) \, dt,
\]
the desired result is immediate.

Q.E.D.